

# On chiral magnetic effect and holography

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## Abstract

We point out that there is a difference between the behavior of fermionic systems (and their holographic analogs) in a background axial vector field, on the one hand, and at finite chiral chemical potential, on the other. In the former case, the electric current induced by constant background axial field  $A_0$  and magnetic field  $\mathbf{B}$  vanishes, while in the latter it is given by the anomaly-prescribed formula  $\mathbf{j} = \frac{\mu_A}{2\pi^2} e^2 N_c \mathbf{B}$ .

## 1 Introduction and summary

The chiral magnetic effect (CME) [1, 2], in its simplest version, is that chirally asymmetric quark matter in background magnetic field  $\mathbf{B}$  develops electric current directed along  $\mathbf{B}$ . The “canonical” expression for the current in a theory with one quark flavor of unit electric charge  $e = 1$  and  $N_c$  colors is [2] (see also Refs. [3, 4, 5, 6, 7, 8] for related discussion)

$$\mathbf{j} = \frac{\mu_A}{2\pi^2} N_c \mathbf{B} , \quad (1)$$

where  $\mu_A$  is the chemical potential to the chiral charge. There is some debate on whether this result is subject to strong interaction corrections [9, 10, 11, 12, 13, 14], and, in particular, whether it holds in holographic models of QCD [11, 12, 13]. The purpose of this note is to point out that Refs. [11, 12, 13] study, in fact, quite a different situation than that relevant for CME. In effect, they consider the electric current induced by the joint action of the background magnetic field  $\mathbf{B}$  and *background temporal component of an axial vector field*  $A_\mu^A$  (the latter field couples to the chiral current). Furthermore, they require that when both electromagnetic (vector) field  $A_\mu^V(x)$  and axial vector field  $A_\mu^A(x)$  are present, the theory remains invariant under electromagnetic gauge transformations. The latter requirement yields the Bardeen counterterm [15] that contributes to the expression for the electric current<sup>1</sup>.

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<sup>1</sup>Ref. [13] proceeds by adding a contribution to the electric current coming from the scalar sector of the holographic model, and arrives at several expressions, one of which coincides with (1). Ref. [16] claims that the result (1) is obtained for canonical ensemble, while the electric current calculated for grand canonical ensemble vanishes.

Our observation is twofold. First the chemical potential is *not* the same thing as the static and homogeneous limit of the background axial vector field. Hence, the requirement of electromagnetic gauge invariance in background fields  $A_\mu^V(x)$  and  $A_\mu^A(x)$  is irrelevant; in particular, the Bardeen counterterm is of no direct significance. Second, the chemical potential can be introduced to a *conserved* quantum number only. In the CME case, this is a suitably defined conserved axial charge. Unlike the conserved axial current, which is not invariant under electromagnetic gauge transformations, the conserved axial charge is well defined, as it is invariant under spatially localized electromagnetic gauge transformations<sup>2</sup>.

We will see that once our observation is accounted for, the result (1) is back. This must be the case, as the formula (1) is a direct consequence of the triangle anomaly, as we argue in the end of this note (cf. Ref. [8]).

This note is organized as follows. In section 2 we reproduce the result of Ref. [12] (see also Refs. [11, 13]) in a simple ads/QCD-like setup. The reader may safely skip this section over: the result is of general nature and is easily understood, as we discuss in the beginning of section 3. We then introduce the chemical potential to the conserved axial charge and reproduce the result (1).

## 2 Background vector and axial vector fields

To illustrate our points, let us consider the simplest adS/QCD setup, namely,  $U(1)$  gauge theory in 5 dimensions on an interval  $x^5 \equiv z \in (0, L)$  [17]. We will argue in section 3 that the final result is, in fact, model-independent. We begin with the action

$$S = \int d^4x dz \left( -\frac{1}{4g^2} F_{MN} F_{MN} - \kappa \epsilon^{MNPQR} A_M F_{NP} F_{QR} \right)$$

where  $M, N = 0, 1, 2, 3, 5$ ,  $g$  is 5-dimensional coupling and

$$\kappa = \frac{N_c}{24\pi^2} .$$

The bulk field equation reads

$$\frac{1}{g^2} \partial_N F_{NM} - 3\kappa \epsilon^{MNPQR} F_{NP} F_{QR} = 0 . \quad (2)$$

In particular, for  $M = 5$  one has

$$\frac{1}{g^2} \partial_\mu F_{\mu 5} - 3\kappa \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = 0 . \quad (3)$$

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<sup>2</sup>We leave aside genuine non-conservation of chiral charge due to triangle anomaly involving color gauge fields. The latter is treated separately in the analysis of CME.

Even though the behavior of this system in the background vector and axial vector fields is not directly relevant for CME, let us discuss this behavior to make contact with Refs. [11, 12, 13]. To this end, we introduce the background right and left vector fields  $A_\mu^{R,L}(x^\nu)$ , or, equivalently, background vector and axial fields  $A_\mu^V = A_\mu^R + A_\mu^L$  and  $A_\mu^A = A_\mu^R - A_\mu^L$ . The background fields are identified with the boundary values of the 5-dimensional field,

$$A_\mu(x^\nu, 0) = A_\mu^L(x^\nu) \quad A_\mu(x^\nu, L) = A_\mu^R(x^\nu) .$$

The currents are obtained, as usual, by varying the action with respect to these fields. Provided the bulk equations (2) are satisfied, the expressions for the vector and axial currents are

$$\begin{aligned} j_\mu &= j_\mu^R + j_\mu^L = \frac{1}{g^2} [F_{\mu 5}(z=L) - F_{\mu 5}(z=0)] - 2\kappa\epsilon^{\mu\nu\lambda\rho} (A_\nu^V F_{\lambda\rho}^A + A_\nu^A F_{\lambda\rho}^V) \\ j_\mu^A &= j_\mu^R - j_\mu^L = \frac{1}{g^2} [F_{\mu 5}(z=L) + F_{\mu 5}(z=0)] - 2\kappa\epsilon^{\mu\nu\lambda\rho} (A_\nu^V F_{\lambda\rho}^V + A_\nu^A F_{\lambda\rho}^A) \end{aligned}$$

Note that at this stage, neither vector nor axial current is invariant under the electromagnetic gauge transformations acting on  $A_\mu^V$ .

Making use of the field equation (3), one finds for the divergencies

$$\partial_\mu j_\mu = 2\kappa F_{\mu\nu}^A \tilde{F}_{\mu\nu}^V, \quad \partial_\mu j_\mu^5 = \kappa (F_{\mu\nu}^V \tilde{F}_{\mu\nu}^V + F_{\mu\nu}^A \tilde{F}_{\mu\nu}^A),$$

where  $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$ . This is the same result as in Ref. [12]. Aiming at restoring the conservation of vector current, one adds the Bardeen counterterm into the action,

$$S_B = 2\kappa \int d^4x \epsilon^{\mu\nu\lambda\rho} A_\mu^A A_\nu^V F_{\lambda\rho}^V \quad (4)$$

The corresponding Bardeen currents are

$$j_{\mu B} = -4\kappa\epsilon^{\mu\nu\lambda\rho} A_\nu^A F_{\lambda\rho}^V + 2\kappa\epsilon^{\mu\nu\lambda\rho} A_\nu^V F_{\lambda\rho}^A, \quad j_{\mu,B}^5 = 2\kappa\epsilon^{\mu\nu\lambda\rho} A_\nu^V F_{\lambda\rho}^V.$$

With these terms included, the currents become

$$J_\mu = j_\mu + j_{\mu B} = \frac{1}{g^2} [F_{\mu 5}(z=L) - F_{\mu 5}(z=0)] - 6\kappa\epsilon^{\mu\nu\lambda\rho} A_\nu^A F_{\lambda\rho}^V \quad (5)$$

$$J_\mu^5 = j_\mu^5 + j_{\mu B}^5 = \frac{1}{g^2} [F_{\mu 5}(z=L) + F_{\mu 5}(z=0)] - 2\kappa\epsilon^{\mu\nu\lambda\rho} A_\nu^A F_{\lambda\rho}^A \quad (6)$$

The gauge invariance of the currents under the electromagnetic gauge transformations is now restored, while the divergencies are

$$\partial_\mu J_\mu = 0, \quad \partial_\mu J_\mu^5 = 3\kappa F_{\mu\nu}^V \tilde{F}_{\mu\nu}^V + \kappa F_{\mu\nu}^A \tilde{F}_{\mu\nu}^A \quad (7)$$

This is the standard result.

Finally, let us consider static background with non-vanishing vector-potentials  $A_i^V = A_i^V(\mathbf{x})$ ,  $A_0^A = \text{const}$ , and constant magnetic field  $B_i = \frac{1}{2}\epsilon^{ijk}F_{jk}^V$ . The field equations for static fields  $A_\mu = A_\mu(\mathbf{x}, z)$ ,  $A_5 = 0$  read

$$\frac{1}{g^2}\partial_5^2 A_i - 12\kappa\epsilon^{ijk}\partial_5 A_0 F_{jk} = 0 \quad (8)$$

$$\frac{1}{g^2}\partial_5^2 A_0 + 12\kappa\epsilon^{ijk}\partial_5 A_i F_{jk} = 0 \quad (9)$$

$$\partial_5 \partial_i A_i = 0 \quad (10)$$

This relevant solution to system has the form

$$A_i(\mathbf{x}, z) = \frac{1}{2}A_i^V(\mathbf{x}) + C_i(z), \quad A_0 = A_0(z)$$

with boundary conditions  $C_i(L) = C_i(0) = 0$ ,  $A_0(L) = \frac{1}{2}A_0^A$ ,  $A_0(0) = -\frac{1}{2}A_0^A$ . The function  $A_0(z)$  is antisymmetric with respect to the point  $L/2$ , and Eq. (9) shows that  $C_i(z)$  is symmetric. We derive from Eq. (8) that

$$\frac{1}{g^2}F_{i5} = -\frac{1}{g^2}\partial_5 C_i = -12\kappa A_0 B_i$$

and hence

$$\frac{1}{g^2}[F_{i5}(L) - F_{i5}(0)] = -12\kappa B_i [A_0(L) - A_0(0)] = -12\kappa B_i A_0^A.$$

This term cancels out the second term in the expression (5) for the electric current  $J_i$ , so the current vanishes in the background field configuration we consider,

$$J_i = 0.$$

This is the result obtained in Ref. [12] (see also Refs. [11, 13]).

### 3 Axial chemical potential

The fact that the electromagnetic current vanishes in the background of constant axial vector potential  $A_0^A$  and magnetic field  $\mathbf{B}$ , at least to the first order in  $\mathbf{B}$ , is of general nature. To see this, consider the effective action for the fields  $A_\mu^V$  and  $A_\mu^A$  obtained by integrating out the dynamical degrees of freedom. In terms of it, the electric current is

$$J_i = \frac{\delta S_{eff}[A^V, A^A]}{\delta A_i^V}.$$

If it did not vanish in our background, and had the form  $\mathbf{J} \propto A_0^A \cdot \mathbf{B}$ , then the lowest derivative term in the effective action would have precisely the form of the Bardeen action,

$$S_{eff}[A^V, A^A] = \text{const} \cdot \int d^4x \epsilon^{\mu\nu\lambda\rho} A_\mu^A A_\nu^V F_{\lambda\rho}^V \quad (11)$$

However, the theory, and hence the effective action, is invariant under electromagnetic gauge transformations, so such a term cannot be generated.

Let us now switch off the axial vector field  $A_\mu^A$  and introduce instead finite axial chemical potential  $\mu_A$ . In the first place, the chemical potential is constant in space and time, so the constraints coming from the requirement of the electromagnetic gauge invariance are relaxed. Second, the chemical potential can be introduced to a conserved quantity only. In other words, a quantity well defined for a microcanonical ensemble is a quantum number that does not change when other parameters (like background fields) vary. Precisely because of the anomaly (7), the integral of  $J_0^5$  over space is not conserved. The conserved chiral charge is

$$Q^5 = \int d^3x J_0^5 - 3\kappa \int d^3x \epsilon^{ijk} A_i^V F_{jk}^V$$

Since  $J_0$  is gauge invariant, this chiral charge is invariant under the electromagnetic gauge transformations. Upon adding the chemical potential, the (Euclidean) action of the theory becomes

$$S[\mu] = S - \mu_A \int dx^0 Q^5 = \left( S - \int d^4x \mu_A J_0^5 \right) + 3\kappa \mu_A \int d^4x \epsilon^{ijk} A_i^V F_{jk}^V \quad (12)$$

where  $S$  is the original action of the theory. The dynamical degrees of freedom enter only the term in parenthesis, which is invariant under electromagnetic gauge transformations even for  $\mu_A$  varying in space and time. In other words, when considering the dynamics induced *by this term*, one *can* treat  $\mu_A$  as the static and homogeneous axial vector field (in the model of section 2 this is precisely the dynamics studied there). According to the above argument, this dynamics does not induce the term (11) in the effective action. Hence, the lowest derivative term in the effective action is simply given by the last term in (12) (cf. Ref. [4]),

$$S_{eff} = 3\kappa \mu_A \int d^4x \epsilon^{ijk} A_i^V F_{jk}^V \quad (13)$$

By varying this effective action with respect to  $A_i$ , one arrives at the result (1).

To conclude, any model with correct anomaly structure yields the effective action (13), and hence the expression (1) for the electric current induced in chirally asymmetric matter.

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## References

- [1] D. Kharzeev, Phys. Lett. B **633**, 260 (2006) [arXiv:hep-ph/0406125];  
D. Kharzeev and A. Zhitnitsky, Nucl. Phys. A **797**, 67 (2007) [arXiv:0706.1026 [hep-ph]];  
D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A **803**, 227 (2008) [arXiv:0711.0950 [hep-ph]].
- [2] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008) [arXiv:0808.3382 [hep-ph]].
- [3] A. N. Redlich and L. C. R. Wijewardhana, Phys. Rev. Lett. **54**, 970 (1985).
- [4] V. A. Rubakov and A. N. Tavkhelidze, Phys. Lett. B **165**, 109 (1985) ;  
V. A. Matveev, V. A. Rubakov, A. N. Tavkhelidze and V. F. Tokarev, Nucl. Phys. B **282**, 700 (1987).
- [5] D. Diakonov and V. Y. Petrov, Phys. Lett. B **275**, 459 (1992).
- [6] A. Y. Alekseev, V. V. Cheianov and J. Frohlich, Phys. Rev. Lett. **81**, 3503 (1998) [arXiv:cond-mat/9803346].
- [7] D. T. Son and A. R. Zhitnitsky, Phys. Rev. D **70**, 074018 (2004) [arXiv:hep-ph/0405216];  
M. A. Metlitski and A. R. Zhitnitsky, Phys. Rev. D **72**, 045011 (2005) [arXiv:hep-ph/0505072].
- [8] G. M. Newman and D. T. Son, Phys. Rev. D **73**, 045006 (2006) [arXiv:hep-ph/0510049].
- [9] O. Bergman, G. Lifschytz and M. Lippert, Phys. Rev. D **79**, 105024 (2009) [arXiv:0806.0366 [hep-th]].
- [10] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya and M. I. Polikarpov, Phys. Rev. D **80**, 054503 (2009) [arXiv:0907.0494 [hep-lat]].
- [11] H. U. Yee, JHEP **0911**, 085 (2009) [arXiv:0908.4189 [hep-th]].
- [12] A. Rebhan, A. Schmitt and S. A. Stricker, JHEP **1001**, 026 (2010) [arXiv:0909.4782 [hep-th]].
- [13] A. Gorsky, P. N. Kopnin and A. V. Zayakin, arXiv:1003.2293 [hep-ph].
- [14] K. Fukushima and M. Ruggieri, arXiv:1004.2769 [hep-ph].
- [15] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

- [16] H. U. Yee, talk given at Workshop on P- and CP-odd Effects in Hot and Dense Matter, Brookhaven, April 26 – 30, 2010; <http://quark.phy.bnl.gov/~kharzeev/cpodd/yee.pdf>
- [17] D. T. Son and M. A. Stephanov, Phys. Rev. D **69**, 065020 (2004) [arXiv:hep-ph/0304182].